Introduction

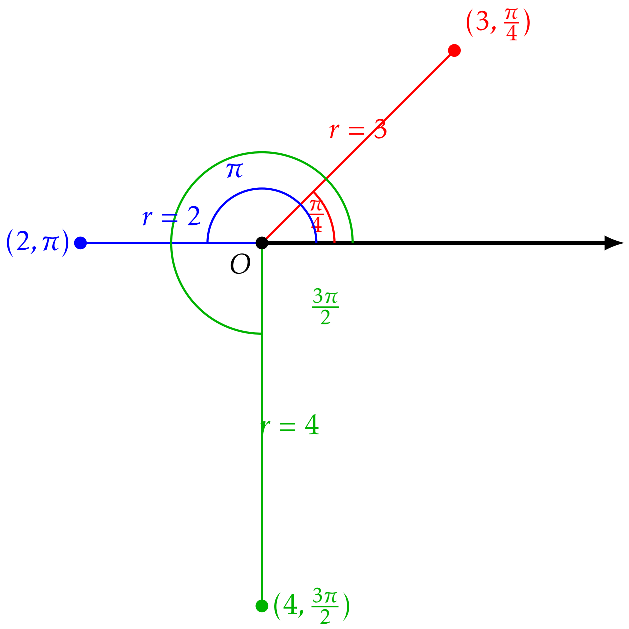
Polar coordinates provide an alternative way to describe the position of points in the plane. Instead of using horizontal and vertical distances (as in Cartesian coordinates), polar coordinates use distance from a fixed point and an angle.

Fact — A point in polar coordinates is described by (r, θ) where:

- r is the distance from the origin (pole) to the point
- θ is the angle measured anticlockwise from the positive x-axis (initial line)

Example

Plot the points with polar coordinates $(3, \frac{\pi}{4})$, $(2, \pi)$, and $(4, \frac{3\pi}{2})$.



Converting Between Coordinate Systems

Fact (Polar to Cartesian) — If a point has polar coordinates (r, θ), then its Cartesian coordinates are:

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}$$

Fact (Cartesian to Polar) — If a point has Cartesian coordinates (x, y), then its polar coordinates are:

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ \tan\theta &= \frac{y}{x} \quad \text{(being careful about quadrants)} \end{aligned}$$

Example

Convert the point with Cartesian coordinates $(3, -4)$ to polar coordinates.

$$\begin{aligned} r &= \sqrt{3^2 + (-4)^2} = \sqrt{9 + 16} = 5 \\ \tan\theta &= \frac{-4}{3} \end{aligned}$$

Since the point is in the fourth quadrant, $\theta = \arctan\left(\frac{-4}{3}\right) = -0.927$ *radians.*
Therefore, the polar coordinates are $(5, -0.927)$ *or equivalently* $(5, 5.356)$ *using the* 2π*-convention.*

Tip

There are multiple different polar coordinates which refer to the same point. To keep things clearer, we typical restrict the range of $0 \leq \theta < 2\pi$. Also the point $(0, \theta)$

Example

Convert the point with polar coordinates $(6, \frac{2\pi}{3})$ to Cartesian coordinates.

$$\begin{aligned} x &= 6\cos\left(\frac{2\pi}{3}\right) = 6 \times \left(-\frac{1}{2}\right) = -3 \\ y &= 6\sin\left(\frac{2\pi}{3}\right) = 6 \times \frac{\sqrt{3}}{2} = 3\sqrt{3} \end{aligned}$$

Therefore, the Cartesian coordinates are $(-3, 3\sqrt{3})$.

Equations in Polar and Cartesian Forms

Example

Find the polar equations of:

- $(x^2 + y^2)^2 = 4xy$
- $y = x^2$
- $x \cos\alpha + y \sin\alpha = p$

- *Since* $x = r\cos\theta, y = r\sin\theta$, *we have* $r^4 = 4r^2\cos\theta\sin\theta \Rightarrow r^2 = 2\sin 2\theta$.
- *We now have* $r\sin\theta = r^2\cos^2\theta \Rightarrow r = \frac{\sin\theta}{\cos^2\theta} = \sec\theta\tan\theta$
- *Notice that we have* $r\cos\theta\cos\alpha + r\sin\theta\sin\alpha = p \Rightarrow r\cos(\theta - \alpha) = p$

Example

Find the cartesian equations of the curves with the polar coordinates

- $r = 2a\cos\theta$ for $-\frac{1}{2}\pi \leq \theta \leq \frac{1}{2}\pi$
- $r^2 = a^2 \sin 2\theta$

- *Since* $\cos\theta = \frac{x}{r}$ *we have* $r = 2a\frac{x}{r} \Rightarrow r^2 = 2ax \Rightarrow x^2 + y^2 = 2ax \Rightarrow (x-a)^2 + y^2 = a^2$
  Consider a circle radius 2a from the origin and look at the length of r

- $x^2 + y^2 = 2a^2 \cos\theta \sin\theta = 2a^2 \frac{xy}{r^2} \Rightarrow (x^2 + y^2)^2 = 2a^2 xy$. *Notice this means* $xy \geq 0$

Sketching Polar Curves

Fact — A polar equation has the form $r = f(\theta)$ and describes a curve in the plane.

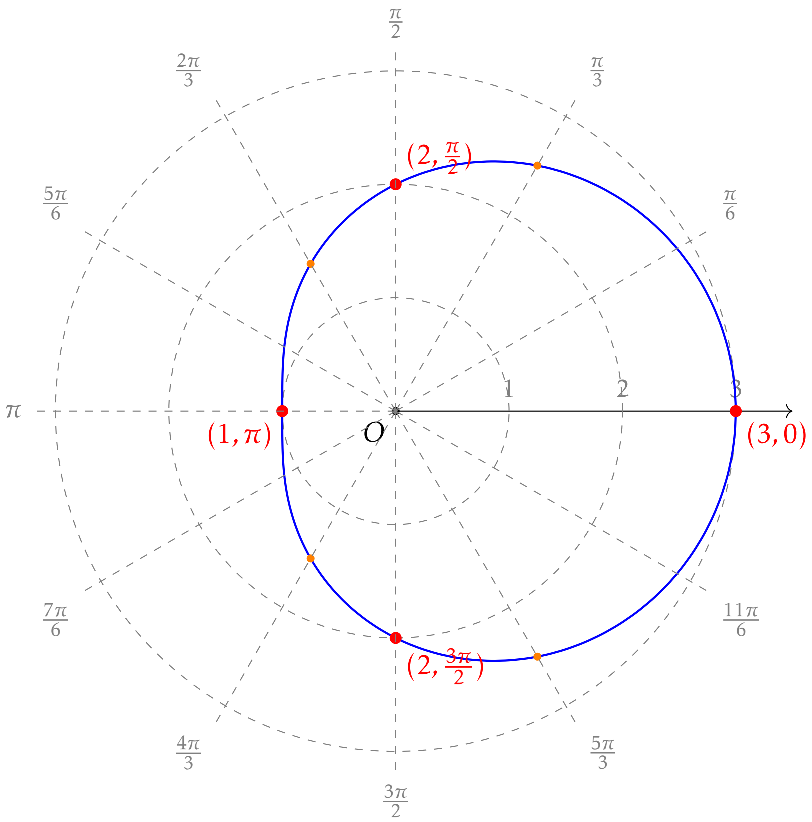
Example

Sketch the graph of $r = 2 + \cos\theta$ for $0 \le \theta \le 2\pi$.

First, create a table of values:

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\cos\theta$	1	0	-1	0	1
r	3	2	1	2	3

The curve is a limacon (snail-shaped curve) that starts at $(3,0)$*, curves inward to a minimum distance of 1 from the origin at* $\theta = \pi$*, then back out.*

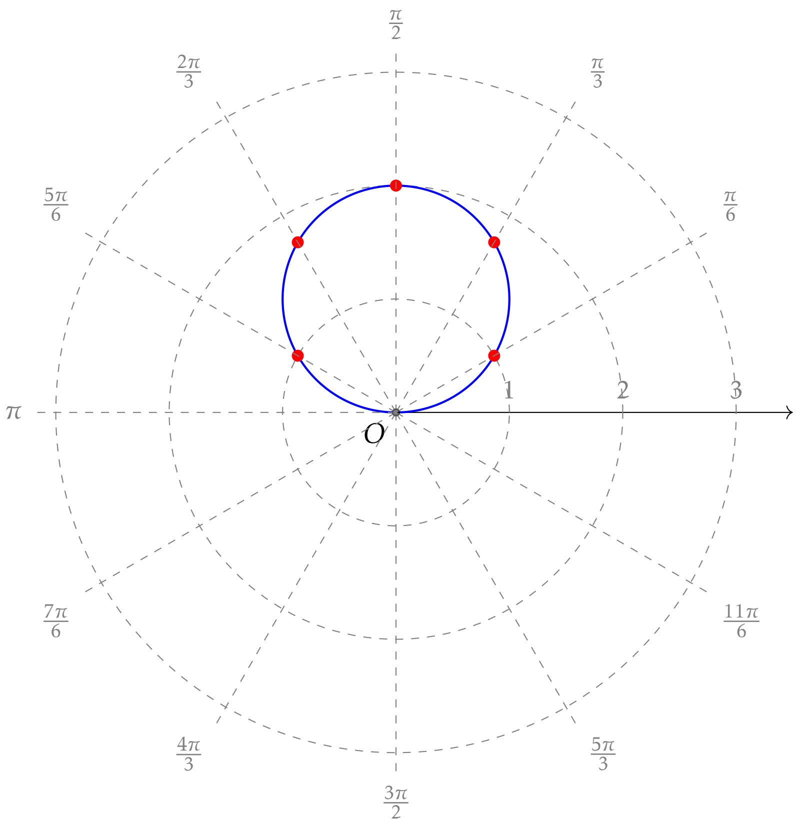


Example

Sketch the graph of $r = 2\sin\theta$ for $0 \leq \theta \leq \pi$.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
r	0	1	$\sqrt{2}$	$\sqrt{3}$	2	$\sqrt{3}$	$\sqrt{2}$	1	0

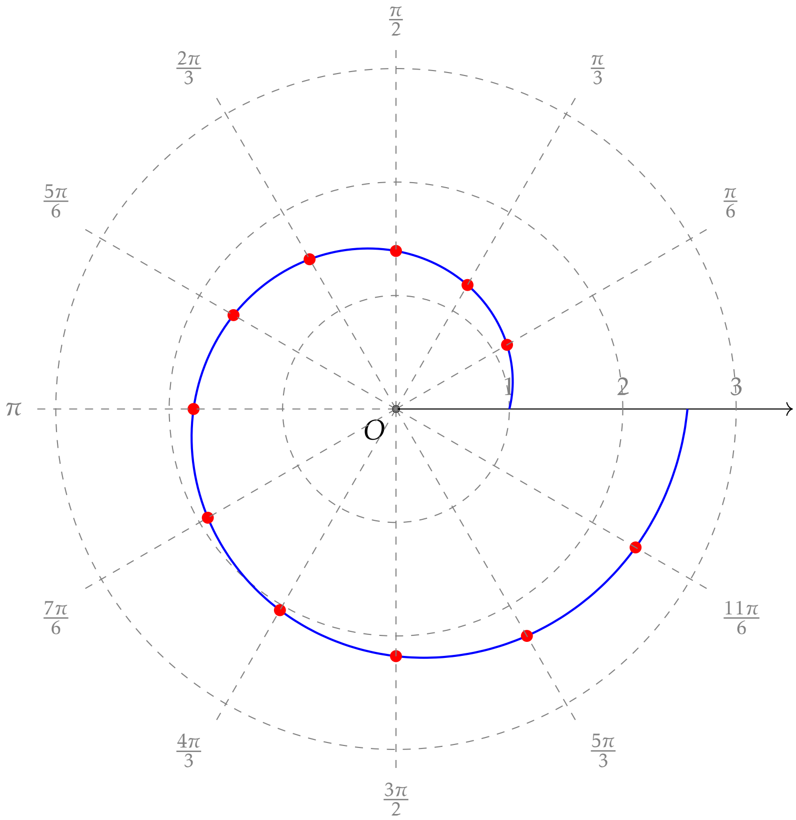
This creates a circle passing through the origin with center at $(0,1)$ in Cartesian coordinates.



Example

Sketch the graph of $r = 1 + \frac{\theta}{4}$ for $0 \leq \theta \leq 2\pi$.

This creates a spiral (and Archimedian spiral)



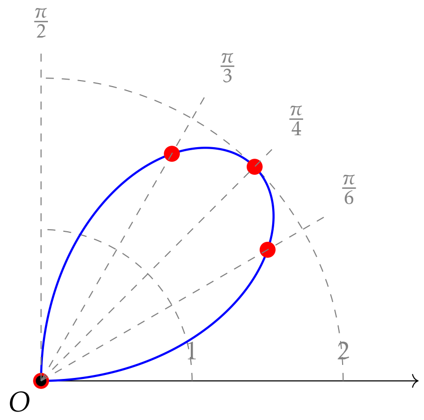
Symmetry in Polar Graphs

Fact (Tests for Symmetry) — For a polar equation $r = f(\theta)$:

- **Symmetry about the x-axis**: If $f(-\theta) \equiv f(\theta)$
- **Symmetry about the y-axis**: If $f(\pi - \theta) \equiv f(\theta)$
- **Symmetry about the origin**: If $f(\theta + \pi) \equiv f(\theta)$
- **Symmetry about the line** $\theta = \alpha$: If $f(2\alpha - \theta) \equiv f(\theta)$

Example

Draw the graph with equation $r = 2\sin 2\theta$ for $0 \le \theta \le \frac{1}{2}\pi$. Prove that it is symmetrical about the line $\theta = \frac{\pi}{4}$



Tip

When sketching polar curves:

1. Check for symmetry first
2. Find where $r = 0$ (curves pass through origin)
3. Find maximum and minimum values of r
4. Plot key points and use symmetry to complete the curve

Greatest and least values of polar curves

> **Example**
>
> For the graph with polar equation $r = 3 + 2\cos 3\theta$ with $-\pi < \theta \leq \pi$, find the greatest and least values of r, and the values of for which they occur

Since $\cos$ *ranges from* -1 *to* 1 *over this range,* r *ranges from* 1 *to* 5*, with minimums when* $\cos 3\theta = -1 \Rightarrow 3\theta = (2n+1)\pi \Rightarrow \theta = -\frac{\pi}{3}, \frac{\pi}{3}, \pi$ *and maximums when* $\cos 3\theta = 1 \Rightarrow 3\theta = 2n\pi \Rightarrow \theta = -\frac{2\pi}{3}, 0, \frac{2\pi}{3}$

> **Example**
>
> Find the maximum and minimum values of r for the curve with polar equation $r = 2 + \cos\theta + \cos 2\theta$ for $-\pi < \theta \leq \theta$.

$$\begin{aligned}\frac{\mathrm{d}r}{\mathrm{d}\theta} &= -\sin\theta - 2\sin 2\theta \\ &= -\sin\theta - 4\sin\theta\cos\theta \\ &= -\sin\theta(1 + 4\cos\theta)\end{aligned}$$

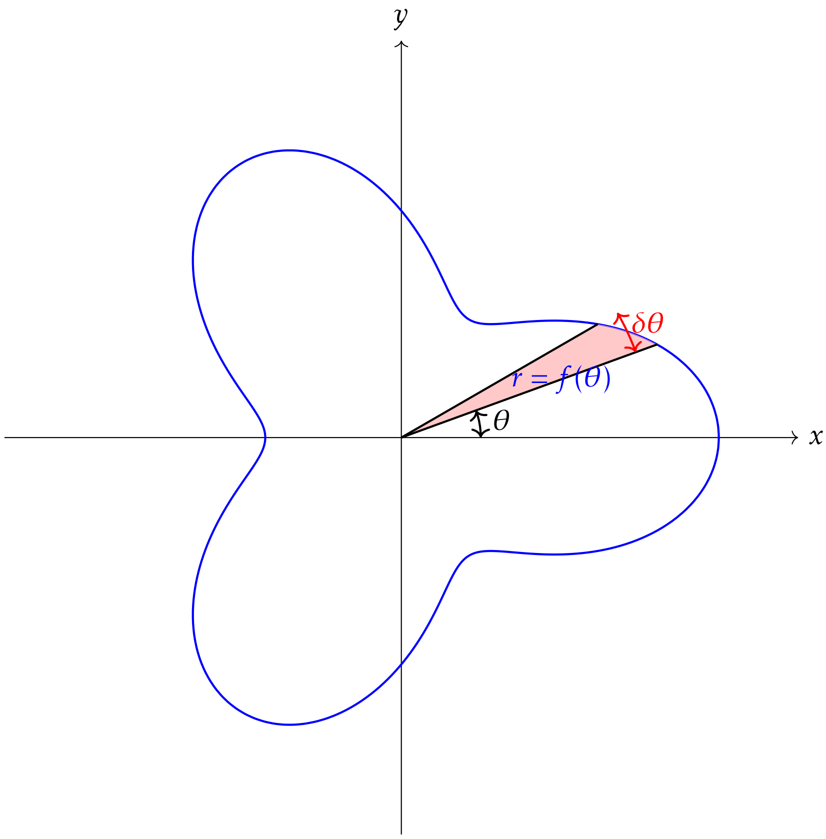
Therefore $\sin\theta = 0$ *or* $\cos\theta = -\frac{1}{4}$.

$$\frac{\mathrm{d}^2 r}{\mathrm{d}\theta^2} = -\cos\theta - 4\cos 2\theta$$

When $\theta = 0$, $\frac{\mathrm{d}^2 r}{\mathrm{d}\theta^2} = -5 \Rightarrow$ *max When* $\theta = \pi$, $\frac{\mathrm{d}^2 r}{\mathrm{d}\theta^2} = -3 \Rightarrow$ *max*

When $\cos\theta = -\frac{1}{4}$, $\frac{\mathrm{d}^2 r}{\mathrm{d}\theta^2} = -(-\frac{1}{4}) - 4(-\frac{7}{8}) = \frac{15}{4} \Rightarrow$ *min*

Areas of Polar Curves



The area of the wedge is $\frac{1}{2}r^2\delta\theta = \frac{1}{2}[f(\theta)]^2\delta\theta$*, so summing over all the deltas.*

Fact (Area Formula) — The area enclosed by a polar curve $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$ is:

$$A = \frac{1}{2}\int_{\alpha}^{\beta} r^2\,d\theta = \frac{1}{2}\int_{\alpha}^{\beta} [f(\theta)]^2\,d\theta$$

Example

Find the area enclosed by the circle $r = 4$.

$$\begin{aligned} A &= \frac{1}{2}\int_0^{2\pi} r^2\,d\theta \\ &= \frac{1}{2}\int_0^{2\pi} 16\,d\theta \\ &= \frac{1}{2}\times 16\times 2\pi \\ &= 16\pi \end{aligned}$$

This matches the expected result for a circle of radius 4: $\pi r^2 = \pi\times 4^2 = 16\pi$.

Example

Find the area enclosed by one petal of the rose curve $r = 3\sin(2\theta)$.

One petal occurs when $0 \leq \theta \leq \frac{\pi}{2}$ *(when* $\sin(2\theta) \geq 0$*).*

$$\begin{aligned} A &= \frac{1}{2}\int_0^{\pi/2} [3\sin(2\theta)]^2 \, d\theta \\ &= \frac{1}{2}\int_0^{\pi/2} 9\sin^2(2\theta) \, d\theta \\ &= \frac{9}{2}\int_0^{\pi/2} \sin^2(2\theta) \, d\theta \end{aligned}$$

Using the identity $\sin^2(2\theta) = \frac{1-\cos(4\theta)}{2}$*:*

$$\begin{aligned} A &= \frac{9}{2}\int_0^{\pi/2} \frac{1-\cos(4\theta)}{2} \, d\theta \\ &= \frac{9}{4}\int_0^{\pi/2} [1-\cos(4\theta)] \, d\theta \\ &= \frac{9}{4}\left[\theta - \frac{\sin(4\theta)}{4}\right]_0^{\pi/2} \\ &= \frac{9}{4}\left[\frac{\pi}{2} - 0\right] = \frac{9\pi}{8} \end{aligned}$$